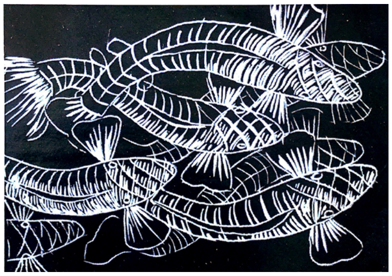
Are Induction and Well-Ordering Equivalent? II

Lars–Daniel Öhman



Nota. *Estimados lectores reproducimos a continuación la segunda parte del artículo* Are Induction and Well-Ordering Equivalent?*, escrito por el profesor Lars–Daniel Öhman.*
La primera parte la pueden consultar en el número 702 del Boletín.
Ya el título provoca curiosidad y una microscópica angustia. ¿El principio de inducción y el principio del buen orden son, en verdad, equivalentes?
Resulta que esta equivalencia, que uno creía superada, una vez que uno pasaba por los cursos básicos de álgebra, tiene detalles que vale la pena estudiar nuevamente.
En términos generales la respuesta es: No, no son equivalentes. Cómo es esto posible es el tema central del texto del profesor Lars–Daniel Öhman.
Al parecer parte del atractivo de las licenciaturas que se imparten en nuestra Facultad es la sensación de estar rodeado de misterios, de preguntas que nadie ha podido resolver, de estar expuestos a nuevas propuestas que nos "mueven el piso".
A este ambiente hay que agregar aquellas respuestas que parecían cerrar un asunto definitivamente, pero que resulta que no, que no todo está dicho. Que gracias a la obsesión de muchos colegas, estudiantes y profesores, las respuestas vuelven a ser estudiadas y, oh sorpresa, hay detalles bien interesantes que se nos escaparon.
Agradecemos a las profesoras Gaby Campero y Pilar Valencia el llamar nuestra atención hacia este tema.
La referencia completa es esta:

Are Induction and Well-Ordering Equivalent?.
Öhman, Lars–Daniel (6 May 2019). The Mathematical Intelligencer. 41(3): pages 33–40.

La versión completa del artículo se puede consultar en este enlace:

https://rdcu.be/cAssP

Ojalá que disfruten este texto.

Peano, Induction, Well Ordering, and Equivalence

In the late nineteenth century, Giuseppe Peano was thinking about a set N together with a function S:N→N and a certain object 0, with the properties that:

1. 0 belongs to N;
2. If n∈*N*, then S(n)∈*N*;
3. S(n)≠0 for every n∈*N*;
4. If S(n)=S(m), then n=m;
5. If M is a subset of N such that 0 belongs to *M* and S(m)∈*M* for every m∈*M*, then M=N.

Please note that I do not intend to take a stand on the controversial question whether 0 is a natural number. None of the arguments in the present paper hinge on the inclusion of 0 among the natural numbers.

I have left out the axioms regulating how equality works, but it is reflexive, symmetric, and transitive, as would be expected. Note also that some concepts from set theory are assumed, at the very least the concept of set itself, the fundamental membership relation ∈, and the equality of sets. The fifth property, or axiom, is the axiom of induction, or the induction principle.

Peano's function S is usually called the successor function, and it conveys an order < on the elements of N, by the following rules: for every n∈N, one has n<S(n), and if n<m, then n<S(m).

Now, as far as I can tell from biographical sources, Peano had no children, but if he had had children, he might have asked them what it was that he was thinking about. In light of Dedekind's proof that this set of axioms is categorical, any guess other than (some isomorphic version of) the natural numbers, N, would be wrong. In particular, guessing "the whole numbers" or "the ordinal numbers up to ω + ω" would have been wrong.

Let us suppose that as the game went on, Peano would give the same first four clues, but instead of the fifth he would give the clue

(5′): Every nonempty subset *M*⊂*N* has a least member,

where the meaning of "least" is in relation to the order relation < defined on the basis of the function S. This is the well-ordering principle. Certainly, guessing "the natural numbers" could still be correct, since the natural numbers satisfy this property. However, guessing "the ordinal numbers up to ω + ω" could not be refuted as an incorrect guess, since this model also satisfies properties (1)–(4) and (5′), as remarked by Perry.

Expanding on this remark, we denote the set of ordinal numbers up to ω + ω by Ord. Here ω is the standard symbol used for the first limit ordinal, that is, the first ordinal to come after all the natural numbers. A number-line-style illustration of the ordinal numbers up to ω + ω is given in Figure 1.

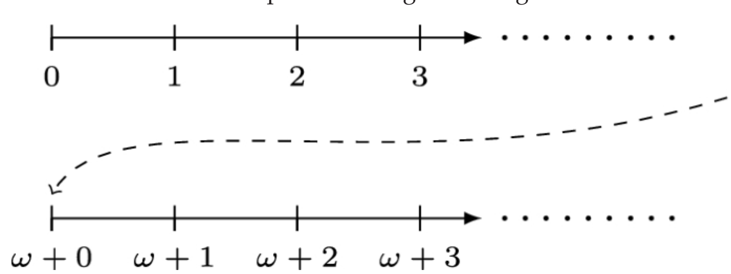


Figure 1

The order relation < on Ord is such that n < ω + m and n<m + ω = ω for all natural numbers n and m, and within each of the number lines, < works as for the natural numbers. Note in particular that addition of ordinals is not commutative.
To see intuitively that every nonempty subset M of Ord has a least member, suppose M contains some ordinal numbers corresponding to ordinary natural numbers (that is, in the upper number line in the figure). Then the least member of M is the least of these natural numbers. If M contains only ordinals of the form ω + n, then the least of them can be found by considering only the natural numbers in the +n part of the ordinals in M. One could say that Ord inherits its well-ordering from the natural numbers, separately for each of the number lines in Figure 1.
In Ord, however, the induction axiom does not hold, since ω is not the successor of any of the previous ordinal numbers, and ω has no immediate predecessor.
In light of the above discussion, we draw two conclusions:

(A) The induction principle and the well-ordering principle are not equivalent relative to axioms (1)–(4) of the Peano system, since the resulting axiomatic systems admit different models.

(B) In the axiomatic system consisting of axioms (1)–(4) together with axiom (5′), induction (5) can't be a theorem, since there is a model Ord in which all these five axioms are satisfied, but induction (5) is not true.

We may also note that the axiomatic system consisting of axioms (1)–(5) admits only models that are isomorphic to the natural numbers, and since the natural numbers are well-ordered, in this system well-ordering (5′) is in fact a theorem. Induction (5) is therefore stronger than well-ordering (5′) in this context, in that it has the power to rule out more possible models.

What Goes Wrong?
In the sources I have looked at that "prove" from axioms (1)–(4) and (5′) that (5) holds, there is a common unjustified step of the proof, namely that every $n \in N$ has a unique immediate predecessor (perhaps denoted by $n-1$). This property, however, does not follow from axioms (1)–(4) and (5′), as evidenced by the existence of a model Ord in which this property does not hold. Specifically, the limit ordinal ω, for example, has no immediate predecessor.

How the Misconceptions Have Spread
It seems natural to assume that the wide diffusion of the imprecise claim that the induction principle and the well-ordering principle are equivalent has been facilitated mainly by its inclusion in widely used textbooks. Specialized sources in axiomatics and set theory do not seem to make the mistake. Rather, it is the sources that treat the axiomatic introduction of the natural numbers in a cursory fashion, as preliminaries to some other subject, that seem most likely to be sketchy on the details.
As mentioned above, I have myself been guilty of repeating this sketchiness in a textbook, so using introspection to analyze the mechanisms of how the misconception has spread indicates that doctrine (by which I mean knowledge spread through teaching) plays a central role. I think it would be most interesting to see a more thorough historical investigation into these issues. Additionally, I have searched for, but not found, some source giving a more detailed overview of alternative ways of introducing and characterizing the natural numbers, perhaps also including an analysis of the relative strength of some different selections of axioms.

Sobre nuestra portada

Frank Hyder (American, b. 1951) is an established name in the contemporary art world.
The Artist has participated in more than 150 group shows and has had over 80 solo exhibitions throughout North, South and Central America, including 8 individual exhibitions in New York City.
He has been one of the few North Americans to have solo museum exhibitions in Venezuela at the Museo de Arte Contemporáneo de Caracas, Museo Jacobo Borges, Museo de Arte Contemporáneo Zulia, Museo Universidad de Los Andes and Museo de Arte Contemporáneo de Coro.

55th Spring Topology and Dynamical Systems Conference

We are pleased to announce that Baylor University will be hosting the 55th Spring Topology and Dynamical Systems Conference from the afternoon of Wednesday, March 9th, through the morning of Sunday, March 13, 2022.
The conference will be held in-person in the Baylor Science Building on the campus of Baylor University in Waco, TX. We will be hosting a workshop on Topological Methods in Dynamical Systems the afternoon of March 9th, followed by a welcome reception. Primary conference activities will begin on Thursday, March 10th. Further information can be found on the conference website,

https://sites.baylor.edu/topology-conference/

as it becomes available.
The 55th STDC will feature five special sessions: Continuum Theory, Dynamical Systems, Geometric Group Theory, Geometric Topology, and Set-Theoretic Topology.

Confirmed plenary and semi-plenary speakers for the conference include:
Dror Bar-Natan, University of Toronto
Noel Brady, University of Oklahoma
Michael Hrusak, UNAM
Alejandro Illanes, UNAM
Tamara Kucherenko, CUNY
Hector Barriga-Acosta,
University of North Carolina-Charlotte
Dana Bartosova, University of Florida
Jernej Činč, University of Ostrava
Lvzhou Chen, University of Texas
James Farre, Yale University
Benjamin Vejnar, Charles University
Kasia Jankiewicz, University of California-Santa Cruz
Daria Michalik, Jan Kochanowski University
Emily Stark, Wesleyan University
Jennifer Wilson, University of Michigan

For general questions concerning the 55th STDC, contact the local organizers at
stdc.organizers@gmail.com
(please include "STDC" in your email subject line).
For questions regarding special sessions, please contact the appropriate session organizers (contact info available on the conference website).
We look forward to seeing everyone in Waco in March!

COLOQUIO DE LA SOCIEDAD MATEMÁTICA MEXICANA

Biología evolutiva a través de los ojos de ciencias de datos

Dra Claudia Solis-Lemus
Wisconsin Institute for Discovery and Department of Plant Pathology, UW–Madison

Resumen. *Métodos para estimar redes filogenéticas que representan el árbol de la vida expandido con ramas de hibridación son indispensables para la biología evolutiva del siglo XXI. La inferencia de árboles filogenéticos está bien establecida, pero métodos para estimar redes filogenéticas que apenas están en desarrollo. Además, demostrar si la discordancia en árboles de genes puede ser explicada en su totalidad por el modelo de coalescencia en árboles o si es necesario invocar eventos reticulares representa dificultades teóricas y computacionales. En esta plática, abordaremos ambos problemas a través de un método estadístico de pseudo-verosimilitud para estimar redes filogenéticas a partir de secuencias de ADN.*
Al final de la plática comentamos sobre retos estadísticos, matemáticos y computacionales en filogenética.

Jueves 25 de noviembre,
16:30 horas (CDMX)

Suscríbete a nuestro canal de youtube y recibe notificación de este y más eventos: ***@smm_oficial***

Seminario DiferenciaHable

Espacios Moduli de métricas planas

Dra. Ana Karla García Pérez,
Departamento Matemáticas,
Facultad de Ciencias, UNAM

Resumen. *En la plática se hablará sobre variedades diferenciables planas cerradas, las cuales están relacionadas con cierto tipo de grupos, llamados grupos de Bieberbach. A partir de estos grupos se puede dar una descripción de los espacios moduli de métricas planas.*

Jueves 25 de noviembre de 2021,
de 12:00 a 13:00 hrs.

Enlace de meet para la reunión:

https://meet.google.com/qsp-gffa-wzz

Eugenio Garnica y Federico Sánchez B.